INSTANTON COUNTING VIA AFFINE LIE ALGEBRAS I: EQUIVARIANT J-FUNCTIONS OF (AFFINE) FLAG MANIFOLDS AND WHITTAKER VECTORS

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ABSTRACT. Let $\mathfrak{g}$ be a simple complex Lie algebra, G - the corresponding simply connected group; let also $\mathfrak{g}_{\mathrm{aff}}$ be the corresponding untwisted affine Lie algebra. For a parabolic subgroup $P \subset G$ we introduce a generating function $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ which roughly speaking counts framed G-bundles on $\mathbb{P}^2$ endowed with a P-structure on the horizontal line (the formal definition uses the corresponding Uhlenbeck type compactifications studied in [3]). In the case $P = G$ the function $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ coincides with Nekrasov's partition function introduced in [23] and studied thoroughly in [24] and [22] for $G = SL(n)$. In the "opposite case" when P is a Borel subgroup of G we show that $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ is equal (roughly speaking) to the Whittaker matrix coefficient in the universal Verma module for the Lie algebra $\check{\mathfrak{g}}_{\mathrm{aff}}$ – the Langlands dual Lie algebra of $\mathfrak{g}_{\mathrm{aff}}$. This clarifies somewhat the connection between certain asymptotic of $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ (studied in *loc. cit.* for $P = G$) and the classical affine Toda system. We also explain why the above result gives rise to a calculation of (not yet rigorously defined) equivariant quantum cohomology ring of the affine flag manifold associated with G. In particular, we reprove the results of [13] and [18] about quantum cohomology of ordinary flag manifolds using methods which are totally different from *loc. cit.*

We shall show in a subsequent publication how this allows one to connect certain asymptotic of the function $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ with *the Seiberg-Witten prepotential* (cf. [2], thus proving the main conjecture of [23] for an arbitrary gauge group G (for $G = SL(n)$ it has been proved in [24] and [22] by other methods.

1. INTRODUCTION

1.1. **The partition function.** This paper has grown out of a (still unsuccessful) attempt to understand the following object. Let K be a simply connected connected compact Lie group and let d be a non-negative integer. Denote by $\mathcal{M}_K^d$ the moduli space of (framed) K-instantons on $\mathbb{R}^4$ of second Chern class $-d$. This space can naturally be embedded into a larger *Uhlenbeck space* $\mathcal{U}_K^d$. Both spaces admit a natural action of the group K (by changing the framing at ∞) and the torus $(S^1)^2$ acting on $\mathbb{R}^4$ after choosing an identification $\mathbb{R}^4 \simeq \mathbb{C}^2$. Moreover the maximal torus of $K \times (S^1)^2$ has unique fixed point on $\mathcal{U}_K^d$. Thus we may consider (cf. [23], [22] and Section 2 for precise definitions) the *equivariant integral*

$$\int_{\mathcal{U}_K^d} 1^d$$

of the unit $K \times (S^1)^2$-equivariant cohomology class (which we denote by 1^d) over $\mathcal{U}_K^d$; the integral takes values in the field $\mathcal{K}$ which is the field of fractions of the algebra $\mathcal{A} = H^*_{K \times (S^1)^2}(pt)$ [1]. Note that $\mathcal{A}$ is canonically isomorphic to the algebra of polynomial functions on $\mathfrak{k} \times \mathbb{R}^2$ (here $\mathfrak{k}$ denotes the Lie algebra of K) which are invariant with respect to the adjoint action of K on $\mathfrak{k}$. Thus each $\int_{\mathcal{U}_K^d} 1^d$ may naturally be regarded as a rational function of $a \in \mathfrak{k}$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$.

Consider now the generating function

$$\mathcal{Z} = \sum_{d=0}^{\infty} Q^d \int_{\mathcal{U}_K^d} 1^d.$$

It can (and should) be thought of as a function of the variables $\mathfrak{q}$ and $a, \varepsilon_1, \varepsilon_2$ as before. In [23] it was conjectured that the first term of the asymptotic in the limit $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \ln \mathcal{Z}$ is closely related to *Seiberg-Witten prepotential* of K. For $K = SU(n)$ this conjecture has been proved in [24] and [22]. Also in [23] an explicit combinatorial expression for $\mathcal{Z}$ has been found.

1.2. **Algebraic version.**

In this paper we want to generalize the definition of the function $\mathcal{Z}$ in several directions and compute it in some of these new cases. First of all, it will be convenient for us to make the whole situation completely algebraic.

Namely, let G be a complex semi-simple algebraic group whose maximal compact subgroup is isomorphic to K. We shall denote by $\mathfrak{g}$ its Lie algebra. Let also $\mathbf{S} = \mathbb{P}^2$ and denote by $\mathbf{D}_\infty \subset \mathbf{S}$ the "straight line at ∞"; thus $\mathbf{S} \backslash \mathbf{D}_\infty = \mathbb{A}^2$. It is well-known that $\mathcal{M}_K^d$ is isomorphic to the moduli space $\mathrm{Bun}_G^d(\mathbf{S}, \mathbf{D}_\infty)$ of principal G-bundles on $\mathbf{S}$ endowed with a trivialization on $\mathbf{D}_\infty$ of second Chern class $-d$. When it does not lead to a confusion we shall write Bun_G instead of $\mathrm{Bun}_G(\mathbf{S}, \mathbf{D}_\infty)$. The algebraic analog of $\mathcal{U}_K^d$ has been constructed in [3]; we denote this algebraic variety by $\mathcal{U}_G^d$. This variety is endowed with a natural action on $G \times (\mathbb{C}^*)^2$.

1.3. **Parabolic generalization of the partition function.**

Let $\mathbf{C} \subset \mathbf{S}$ denote the "horizontal line". Choose a parabolic subgroup $P \subset G$. Let $\mathrm{Bun}_{G,P}$ denote the moduli space of the following objects:

1) A principal G-bundle $\mathcal{F}_G$ on $\mathbf{S}$;
2) A trivialization of $\mathcal{F}_G$ on $\mathbf{D}_\infty \subset \mathbf{S}$;
3) A reduction of $\mathcal{F}_G$ to P on $\mathbf{C}$ compatible with the trivialization of $\mathcal{F}_G$ on $\mathbf{C}$.

Let us describe the connected components of $\mathrm{Bun}_{G,P}$. Let M be the Levi group of P. Denote by $\check{M}$ the *Langlands dual* group of M and let $Z(\check{M})$ be its center. We denote by $\Lambda_{G,P}$ the lattice of characters of $Z(\check{M})$. Let also $\Lambda_{G,P}^{\mathrm{aff}} = \Lambda_{G,P} \times \mathbb{Z}$ be the lattice of characters of $Z(\check{M}) \times \mathbb{C}^*$. Note that $\Lambda_{G,G}^{\mathrm{aff}} = \mathbb{Z}$.

The lattice $\Lambda_{G,P}^{\mathrm{aff}}$ contains canonical semi-group $\Lambda_{G,P}^{\mathrm{aff,pos}}$ of positive elements (cf. [3] and Section 3). It is not difficult to see that the connected components of $\mathrm{Bun}_{G,P}$ are

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[1]In this paper we always consider cohomology with complex coefficients

parameterized by the elements of $\Lambda_{G,P}^{\mathrm{aff,pos}}$:

$$\mathrm{Bun}_{G,P} = \bigcup_{\theta_{\mathrm{aff}} \in \Lambda_{G,P}^{\mathrm{aff,pos}}} \mathrm{Bun}_{G,P}^{\theta_{\mathrm{aff}}}.$$

Typically, for $\theta_{\mathrm{aff}} \in \Lambda_{G,P}^{\mathrm{aff}}$ we shall write $\theta_{\mathrm{aff}} = (\theta, d)$ where $\theta \in \Lambda_{G,P}$ and $d \in \mathbb{Z}$.

Each $\mathrm{Bun}_{G,P}^{\theta_{\mathrm{aff}}}$ is naturally acted on by $P \times (\mathbb{C}^*)^2$; by embedding M into P we get an action of $M \times (\mathbb{C}^*)^2$ on $\mathrm{Bun}_{G,P}^{\theta_{\mathrm{aff}}}$. In [3] we define for each $\theta_{\mathrm{aff}} \in \Lambda_{G,P}^{\mathrm{aff,pos}}$ certain Uhlenbeck scheme $\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}$ which contains $\mathrm{Bun}_{G,P}^{\theta_{\mathrm{aff}}}$ as a dense open subset. The scheme $\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}$ still admits an action of $M \times (\mathbb{C}^*)^2$.

We want to do some equivariant intersection theory on the spaces $\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}$. For this let us denote by $\mathcal{A}_{M \times (\mathbb{C}^*)^2}$ the algebra $H_{M \times (\mathbb{C}^*)^2}^*(pt, \mathbb{C})$. Of course this is just the algebra of M-invariant polynomials on $\mathfrak{m} \times \mathbb{C}^2$. Let also $\mathcal{K}_{M \times (\mathbb{C}^*)^2}$ be its field of fractions. We can think about elements of $\mathcal{K}_{M \times (\mathbb{C}^*)^2}$ as rational functions on $\mathfrak{m} \times \mathbb{C}^2$ which are invariant with respect to the adjoint action.

Let $T \subset M$ be a maximal torus. Then one can show that $(\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}})^{T \times (\mathbb{C}^*)^2}$ consists of one point. This guarantees that we may consider the integral $\int_{\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}} 1_{G,P}^{\theta_{\mathrm{aff}}}$ where $1_{G,P}^{\theta_{\mathrm{aff}}}$ denotes the unit class in $H_{M \times (\mathbb{C}^*)^2}^*(\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}, \mathbb{C})$. The result can be thought of as a rational function on $\mathfrak{m} \times \mathbb{C}^2$ which is invariant with respect to the adjoint action of M. Define

$$\mathcal{Z}_{G,P}^{\mathrm{aff}} = \sum_{\theta \in \Lambda_{G,P}^{\mathrm{aff}}} \mathfrak{q}_{\mathrm{aff}}^{\theta_{\mathrm{aff}}} \int_{\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}} 1_{G,P}^{\theta_{\mathrm{aff}}}. \tag{1.1}$$

One should think of $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ as a formal power series in $\mathfrak{q}_{\mathrm{aff}} \in Z(\check{M}) \times \mathbb{C}^*$ with values in the space of ad-invariant rational functions on $\mathfrak{m} \times \mathbb{C}^2$. Typically, we shall write $\mathfrak{q}_{\mathrm{aff}} = (\mathfrak{q}, Q)$ where $\mathfrak{q} \in Z(\check{M})$ and $Q \in \mathbb{C}^*$. Also we shall denote an element of $\mathfrak{m} \times \mathbb{C}^2$ by $(a, \varepsilon_1, \varepsilon_2)$ or (sometimes it will be more convenient) by $(a, \hbar, \varepsilon)$ (note that for general P (unlike in the case $P = G$) the function $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ is not symmetric with respect to switching ε_1 and ε_2).

1.4. **Interpretation via maps and the "finite-dimensional" analog.** Choose now another smooth projective curve $\mathbf{X}$ of genus 0 and with two marked points $0_{\mathbf{X}}, \infty_{\mathbf{X}}$. Choose also a coordinate x on $\mathbf{X}$ such that $x(0_{\mathbf{X}}) = 0$ and $x(\infty_{\mathbf{X}}) = 0$. Let us denote by $\mathcal{G}_{G,P,\mathbf{X}}$ the scheme classifying triples $(\mathcal{F}_G, \beta, \gamma)$, where
   1) $\mathcal{F}_G$ is a principal G-bundle on $\mathbf{X}$;
   2) β is a trivialization of $\mathcal{F}_G$ on the formal neighborhood of $\infty_{\mathbf{X}}$;
   3) γ is a reduction to P of the fiber of $\mathcal{F}_G$ at $0_{\mathbf{X}}$.
   We shall usually omit $\mathbf{X}$ from the notations. We shall also write $\mathcal{G}_G^{\mathrm{aff}}$ for $\mathcal{G}_{G,G}^{\mathrm{aff}}$.

Let $e_{G,P}^{\mathrm{aff}} \in \mathcal{G}_{G,P}^{\mathrm{aff}}$ denote the point corresponding to the trivial $\mathcal{F}_G$ with the natural β and γ. It is explained in [3] that the variety $\mathrm{Bun}_{G,P}$ is canonically isomorphic to

the scheme classifying *based maps* from $(\mathbf{C}, \infty_{\mathbf{C}})$ to $(\mathcal{G}_{G,P}^{\mathrm{aff}}, e_{G,P}^{\mathrm{aff}})$ (i.e. maps from $\mathbf{C}$ to $\mathcal{G}_{G,P}$ sending $\infty_{\mathbf{C}}$ to $e_{G,P}^{\mathrm{aff}}$).

The scheme $\mathcal{G}_{G,P}^{\mathrm{aff}}$ may (and should) be thought of as a partial flag variety for $\mathfrak{g}_{\mathrm{aff}}$. Thus we may consider the following "finite-dimensional" analog of the above problem: for G and P as above let $\mathcal{G}_{G,P} = G/P$. Let also $e_{G,P} \in \mathcal{G}_{G,P}$ denote the image of $e \in G$. Clearly $e_{G,P}$ is stable under the action of P on $\mathcal{G}_{G,P}$. Let ${}^{\mathrm{b}}\mathcal{M}_{G,P}$ denote the moduli space of based maps from $(\mathbf{C}, \infty_{\mathbf{C}})$ to $(\mathcal{G}_{G,P}, e_{G,P})$, i.e. the moduli space of maps $\mathbf{C} \to \mathcal{G}_{G,P}$ which send $\infty_{\mathbf{C}}$ to $e_{G,P}$. This space is acted on by the group $M \times \mathbb{C}^*$. Also ${}^{\mathrm{b}}\mathcal{M}_{G,P}$ is a union of connected components ${}^{\mathrm{b}}\mathcal{M}_{G,P}^{\theta}$ where θ lies in a certain sub-semi-group $\Lambda_{G,P}^{\mathrm{pos}}$ of $\Lambda_{G,P}$. For each θ as above one can also consider the space of *based quasi-maps* (or *Zastava space* in the terminology of [4], [11] and [8]) which we denote by ${}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\theta}$. This is an affine algebraic variety containing ${}^{\mathrm{b}}\mathcal{M}_{G,P}^{\theta}$ as a dense open subset. We also denote by $\mathcal{M}_{G,P}^{\theta}, \mathcal{QM}_{G,P}^{\theta}$ the corresponding spaces of all (unbased) quasi-maps.

We now introduce the "finite-dimensional" analog of the partition function (1.1). As before let
$$\mathcal{A}_{M \times \mathbb{C}^*} = H_{M \times \mathbb{C}^*}^*(pt, \mathbb{C})$$
and denote by $\mathcal{K}_{M \times \mathbb{C}^*}$ its field of fractions. Let also $1_{G,P}^{\theta}$ denote the unit class in the $M \times \mathbb{C}^*$-equivariant cohomology of ${}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\theta}$. Then we define

$$\mathcal{Z}_{G,P} = \sum_{\theta \in \Lambda_{G,P}^{\theta}} \mathfrak{q}^{\theta} \int_{{}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\theta}} 1_{G,P}^{\theta}. \qquad (1.2)$$

This is a formal series in $q \in Z(\check{M})$ with values in the field $\mathcal{K}_{M \times \mathbb{C}^*}$ of M-invariant rational functions on $\mathfrak{m} \times \mathbb{C}$.

In fact the function $\mathcal{Z}_{G,P}$ is a familiar object in Gromov-Witten theory: we shall see later (Theorem 1.7) that up to a simple factor $\mathcal{Z}_{G,P}$ is the so called *equivariant J-function* of $\mathcal{G}_{G,P}$ (cf. Section 1.6 for the details).

1.5. **The Borel case.** We believe that it should be possible to express the function $\mathcal{Z}_{G,P}$ (resp. the function $\mathcal{Z}_{G,P}^{\mathrm{aff}}$) in terms of representation theory of the Lie algebra $\check{\mathfrak{g}}$ (resp. $\check{\mathfrak{g}}_{\mathrm{aff}}$) – by the definition this is a Lie algebra whose root system is dual to that of $\mathfrak{g}$ (resp. to that of $\mathfrak{g}_{\mathrm{aff}}$). One of the motivations for this comes from the main results of [4] and [3] where such a description is found for the intersection cohomology of the varieties ${}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\theta}$ and $\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}$ (in this paper we are going to adopt an intersection cohomology approach to the partition functions $\mathcal{Z}_{G,P}$ and $\mathcal{Z}_{G,P}^{\mathrm{aff}}$; this is explained carefully in Section 2). One of the main results of this paper gives such a calculation of the functions $\mathcal{Z}_{G,B}$ and $\mathcal{Z}_{G,B}^{\mathrm{aff}}$ where $B \subset G$ is a Borel subgroup of G. Roughly speaking we show that $\mathcal{Z}_{G,B}$ (resp. $\mathcal{Z}_{G,B}^{\mathrm{aff}}$) is equal to *Whittaker matrix coefficient* of the Verma module over $\check{\mathfrak{g}}$ (resp. over $\check{\mathfrak{g}}_{\mathrm{aff}}$) whose lowest weight given by $\frac{a}{\hbar} + \rho$ (resp. $\frac{(a, \varepsilon_1)}{\varepsilon_2} + \rho_{\mathrm{aff}}$ where $a, \hbar, \varepsilon_1$ and ε_2 are as in the previous subsection (here we regard (a, ε_1) as a weight for the dual affine algebra $\check{\mathfrak{g}}_{\mathrm{aff}}$; this is explained carefully in Section 3). The precise formulation of this result is given by Theorem 3.4 and Corollary 3.5.

As a corollary we get that the function $\mathfrak{q}^{\frac{a}{\hbar}}\mathcal{Z}_{G,B}$ is an eigen-function of the *quantum Toda hamiltonians* associated with $\check{\mathfrak{g}}$ with eigen-values determined (in the natural way) by a (we refer the reader to [7] for the definition of (affine) Toda integrable system and its relation with Whittaker functions). In the affine case one can also show that $\mathfrak{q}^{\frac{a}{\hbar}}\mathcal{Z}_{G,B}^{\mathrm{aff}}$ is an eigen-function of a certain differential operator which has order 2 ("non-stationary analog" of the affine quadratic Toda hamiltonian). We shall show how this allows to compute the asymptotics of *all* the functions $\mathcal{Z}_{G,P}^{\mathrm{aff}}$ when $\varepsilon_1, \varepsilon_2 \to 0$ in another publication (cf. [2]).

Let us explain the idea of the proof of the above statement (we shall do it in the "finite-dimensional" (i.e. non-affine) case; in the affine case the proof is similar). First for a scheme Y let $\mathrm{IH}^*(Y)$ denote the intersection cohomology of Y with complex coefficients (similarly if a group L acts on Y we denote by $\mathrm{IH}_L^*(Y)$ the corresponding L-equivariant intersection cohomology). Let now

$$\mathrm{IH}_{G,P}^\theta = \mathrm{IH}_{M\times(\mathbb{C}^*)^2}^* ({}^{\mathrm{b}}\mathcal{QM}_{G,P}^\theta) \underset{\mathcal{A}_{M\times\mathbb{C}^*}}{\otimes} \mathcal{K}_{M\times\mathbb{C}^*}.$$

(it is clear that $\mathrm{IH}_{M\times(\mathbb{C}^*)^2}^* ({}^{\mathrm{b}}\mathcal{QM}_{G,P}^\theta)$ has a natural $\mathcal{A}_{M\times\mathbb{C}^*}$-module structure).

Let also

$$\mathrm{IH}_{G,P} = \bigoplus_{\theta\in\Lambda_{G,P}^{\mathrm{pos}}} \mathrm{IH}_{G,P}^\theta$$

Each $\mathrm{IH}_{G,P}^\theta$ is a finite-dimensional vector space over $\mathcal{K}_{M\times\mathbb{C}^*}$ endowed with a (non-degenerate) Poincaré pairing $\langle\cdot,\cdot\rangle_{G,P}^\theta$ taking values in $\mathcal{K}_{M\times\mathbb{C}^*}$. .

Let us now specialize to the case $P = B$. In Section 5 we show that the Lie algebra $\check{\mathfrak{g}}$ acts naturally on $\mathrm{IH}_{G,B}$. Moreover, this action has the following properties. First of all, let us denote by $\langle\cdot,\cdot\rangle_{G,B}$ the direct sum of the pairings $(-1)^{\langle\theta,\check{\rho}\rangle}\langle\cdot,\cdot\rangle_{G,B}^\theta$. [2]

Recall that the Lie algebra $\check{\mathfrak{g}}$ has its triangular decomposition $\check{\mathfrak{g}} = \check{\mathfrak{n}}_+ \oplus \check{\mathfrak{h}} \oplus \check{\mathfrak{n}}_-$. Let $\kappa : \check{\mathfrak{g}} \to \check{\mathfrak{g}}$ denote the Cartan anti-involution which interchanges $\check{\mathfrak{n}}_+$ and $\check{\mathfrak{n}}_-$ and acts as identity on $\check{\mathfrak{h}}$. For each $\lambda \in \mathfrak{h} = (\check{\mathfrak{h}})^*$ we denote by $M(\lambda)$ the corresponding Verma module with lowest weight λ; this is a module generated by a vector v_λ with (the only) relations

$$t(v_\lambda) = \lambda(t)v_\lambda \quad \text{for } t \in \check{\mathfrak{h}} \text{ and } \quad n(v_\lambda) = 0 \quad \text{for } n \in \check{\mathfrak{n}}_-.$$

Then:

1) $\mathrm{IH}_{G,B}$ (with the above action) becomes isomorphic to $M(\lambda)$.

2) $\mathrm{IH}_{G,B}^\theta \subset \mathrm{IH}_{G,B}$ is the $\frac{a}{\hbar} + \rho + \theta$-weight space of $\mathrm{IH}_{G,B}$.

3) For each $g \in \check{\mathfrak{g}}$ and $v, w \in \mathrm{IH}_{G,B}$ we have

$$\langle g(v), w\rangle_{G,B} = \langle v, \kappa(g)w\rangle_{G,B}.$$

4) The vector $\sum_\theta 1_{G,B}^\theta$ (lying is some completion of $\mathrm{IH}_{G,B}$) is a Whittaker vector (i.e. a $\mathfrak{n}_-$-eigen-vector) for the above action.

It is easy to see that the assertions 1-4 imply the above representation-theoretic description of $\mathcal{Z}_{G,B}$ (for this one has to adopt an "intersection cohomology" point of view at the above equivariant integrals; this is explained in Section 2).

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[2] here $\check{\rho}$ denotes the half-sum of the positive roots of $\mathfrak{g}$

In fact, certain parts of the above statement have been known before: for example in [8] the authors compute the dimensions of the spaces $\text{IH}_{G,B}^{\theta}$ and the answer agrees with 1 and 2 above (in fact that these dimensions are known even if we replace $^{b}\mathcal{QM}_{G,B}^{\theta}$ by any $^{b}\mathcal{QM}_{G,P}^{\theta}$ - cf. [4]). Similarly, the computation of dimensions of $\text{IH}_{G,B}^{\theta_{\text{aff}}}$ (defined analogously to $\text{IH}_{G,B}^{\theta}$) follows from [3]. Also the construction of the $\check{\mathfrak{g}}$-action on $\text{IH}_{G,B}$ is very close to the construction in Section 4 of [8].

1.6. Connection with J-functions and quantum cohomology of flag manifolds.
Let us explain how the above result is connected with the known results on quantum cohomology of flag manifolds.

First, let us recall the general set-up (the main reference is [12]). Let X be any smooth projective variety over $\mathbb{C}$. Assume that X is homogeneous, i.e. that the tangent sheaf of X is generated by its global sections. Then for any $\beta \in H_2(X, \mathbb{Z})$ one may consider the moduli space $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of stable maps from a curve Σ of genus zero with n marked points to X of degree β. For every β as above let ev_β denote the evaluation map map at the marked point from $\overline{\mathcal{M}}_{0,1}(X, \beta)$ to X.

The space $\overline{\mathcal{M}}_{0,1}(X, \beta)$ admits canonical line bundle $\mathcal{L}_\beta$ whose fiber over every stable map $C \to X$ is equal to the cotangent space to C at the marked point. We denote by c_β its first Chern class.

In the above set-up one defines the J-function J^X of X in the following way:

$$J^X(t, \hbar) = e^{\frac{t}{\hbar}} \sum_{\beta \in H_2(X, \mathbb{Z})} e^{\langle t, \beta \rangle} (\text{ev}_\beta)_* \left(\frac{1}{\hbar(c_\beta + \hbar)} \right) \tag{1.3}$$

where $t \in H^2(X, \mathbb{C})$[3]. The function J^X takes values in $H^*(X, \mathbb{C}) \otimes \mathbb{C}((\hbar^{-1}))$ which we may think of as a localization of a completion of of $H_{\mathbb{C}^*}(X, \mathbb{C})$ ($\mathbb{C}^*$ acts trivially on X) over $\mathbb{C}[\hbar] = H_{\mathbb{C}^*}^*(pt)$. It is explained in [12] that by looking at the differential equations satisfied by the function J^X one may compute explicitly the small quantum cohomology ring of X. Putting $q = e^t$ one may (formally) write

$$J^X = \mathfrak{q}^{\frac{1}{\hbar}} \sum_{\beta \in H_2(X, \mathbb{Z})} \mathfrak{q}^{\beta} (\text{ev}_\beta)_* \left(\frac{1}{\hbar(c_\beta + \hbar)} \right)$$

In the case when X is acted on by a reductive group G we may consider the equivariant analog of J^X which we shall denote by J_G^X. In this case the function J_G^X takes values in $H_{G \times \mathbb{C}^*}^*(X, \mathbb{C}) \underset{\mathbb{C}[\hbar]}{\otimes} \mathbb{C}((\hbar^{-1}))$. By looking at the differential equations satisfied by J_G^X one may compute the equivariant small quantum cohomology ring of X.

Assume now that $X = \mathcal{G}_{G,P}$. Note that in this case we have the natural identification

$$H_{G \times \mathbb{C}^*}^*(\mathcal{G}_{G,P}, \mathbb{C}) = H_{M \times \mathbb{C}^*}^*(pt)$$

Note also that $H_2(\mathcal{G}_{G,P}, \mathbb{Z}) = \Lambda_{G,P}$ and $H^2(\mathcal{G}_{G,P}) = \text{Lie}\, Z(\check{M})$.

Taking all these identifications into account we now claim the following

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[3]For general X the definition of the push-forward is somewhat tricky - it uses the so called virtual fundamental class. However, for X of the form $\mathcal{G}_{G,P}$ this difficulty is not present.

Theorem 1.7. *For any G and P as above we have*

$$J_G^{\mathcal{G}_{G,P}} = \mathfrak{q}^{\frac{a}{\hbar}} \mathcal{Z}_{G,P}.$$

Theorem 1.7 is proved in Section 6.

Note that in view of the results announced in the previous section it follows that $J_{\mathcal{G}_{G,B}}^G$ can be interpreted via Whittaker matrix coefficients of Verma modules and in particular it is an eigen-function of the Toda hamiltonians for $\check{\mathfrak{g}}$. The latter result is due to B. Kim (cf. [18]); in fact (the "non-affine" part of) this paper may be viewed as a conceptual explanation of Kim's result.

Theorem 1.7 also allows us to think about $\mathfrak{q}^{\frac{a}{\hbar}} \mathcal{Z}_{G,P}^{\mathrm{aff}}$ as the equivariant (with respect to the loop group) J-function of $\mathcal{G}_{G,P}^{\mathrm{aff}}$. The theory of Gromov-Witten invariants of $\mathcal{G}_{G,P}^{\mathrm{aff}}$ is not yet constructerd. However we see from the above that we have a natural candidate for the notion of J-function of $\mathcal{G}_{G,P}$. In particular, the main result of this paper gives a representation-theoretic interpretation of this J-function. This may be viewed as an explanation of the results of [15] and [19] (where the authors compute the quantum cohomology of $\mathcal{G}_{G,P}$ making some assumptions about its existence and properties).

1.8. **Organization of the paper.** This paper is organized as follows. In Section 2 we fix the notation and recall the basic facts about equivariant integration. In Section 3 we give the precise formulation of the main result of this paper discussed above in Section 1.5 (Theorem 3.4). Section 4 is devoted to an explicit proof of the main result for $G = SL(2)$. In Section 5 we prove Theorem 3.4. Finally, Section 6 is devoted to the proof of Theorem 1.7.

1.9. **Acknowledgments.** We are grateful to N. Nekrasov for his patient explanation of the contents of [23] and [24] and numerous discussions on the subjects. We also thank T. Coates, P. Etingof, M. Finkelberg, D. Gaitsgory, D. Kazhdan, H. Nakajima and A. Okounkov for interesting conversations and help at various stages of this work.

2. EQUIVARIANT INTEGRATION

2.1. Let L be a reductive algebraic group over $\mathbb{C}$. Set $\mathcal{A}_L = H_L^*(pt, \mathbb{C})$. It is well known that $\mathcal{A}_L$ is canonically isomorphic to the algebra of invariant polynomials on the Lie algebra $\mathfrak{l}$ of L. Let also $\mathcal{K}_L$ denote the field of fractions of $\mathcal{A}_L$. One can identify $\mathcal{K}_L$ with the field of invariant rational functions on $\mathfrak{l}$. We shall say that an element $f \in \mathcal{K}_L$ is homogeneous of degree $d \in \mathbb{Z}$ if f is a homogeneous rational function of degree d on $\mathfrak{l}$.

2.2. **Equivariant (co)homology.** Let Y be a scheme of finite type over $\mathbb{C}$ endowed with an action of L. We denote by ω_Y the dualizing complex of Y (in the category of constructible sheaves). We will be interested in the equivariant cohomology $H_L^*(Y, \omega_Y)$ (it is sometimes called the *equivariant Borel-Moore homology* of Y). For any closed L-invariant subscheme $Z \subset Y$ we have the natural map $H_L^*(Z, \omega_Z) \to H_L^*(Y, \omega_Y)$.

We may also consider the ordinary equivariant homology $H_{*,L}(Y, \mathbb{C})$ of Y which is (by definition) equal to $H^*_{c,L}(Y, \omega_Y)$. [4]. In particular we have the natural ("forgetting the supports") morphism $H_{*,L}(Y, \mathbb{C}) \to H^*_L(Y, \omega_Y)$. Also since there is a canonical morphism $\mathbb{C}_Y \to \omega_Y[-2\dim Y]$ (here $\mathbb{C}_Y$ denotes the corresponding constant sheaf on Y) we get a canonical map $H^*_L(Y, \mathbb{C}) \to H^{*-2\dim Y}_L(Y, \omega_Y)$ where $H^*_L(Y, \mathbb{C})$ denotes the usual equivariant cohomology of Y.

In addition we have the natural pairing

$$H_{*,L}(Y, \mathbb{C}) \underset{\mathcal{A}_L}{\otimes} H^*_L(Y, \mathbb{C}) \to \mathcal{A}_L \tag{2.1}$$

(it is clear that all the (co)homologies in question are modules over $\mathcal{A}_L$). In particular, we have the integration functional

$$\int_Y : H_{*,L}(Y, \mathbb{C}) \to \mathcal{A}_L$$

which is defined as the pairing with $1 \in H^*_L(Y, \mathbb{C})$. For any closed L-invariant subscheme Z of Y the diagram

$$\begin{array}{ccc}
H_{*,L}(Z, \omega_Z) & \longrightarrow & H_{*,L}(Y, \omega_Y) \\
{\scriptstyle \int_Z} \downarrow & & \downarrow {\scriptstyle \int_Y} \\
\mathcal{A}_L & \xrightarrow{\ \ \mathrm{id}\ \ } & \mathcal{A}_L
\end{array}$$

is commutative.

2.3. Integration on Borel-Moore homology.

We now want to explain that in certain situations one can extend the above integration functional to the Borel-Moore homology. Assume first that L is a torus and set $Z = Y^L$. This is a closed subscheme of Y. It is well-known that in this case the natural morphism $H^*_L(Z, \omega_Z) = H^*(Z, \omega_Z) \otimes \mathcal{A}_L \to H^*_L(Y, \omega_Y)$ becomes an isomorphism after tensoring with $\mathcal{K}_L$.

Assume now (until the end of this section) that Z is proper. Then $H^*_{c,L}(Z, \omega_Z) \simeq H^*_L(Z, \omega_Z)$ we get the map $H^*_L(Y, \omega_Y) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L \to H_{*,L}(Y, \mathbb{C}) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L$ as the composition

$$H^*_L(Y, \omega_Y) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L \simeq H^*_L(Z, \omega_Z) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L \simeq H^*_{c,L}(Z, \omega_Z) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L \to H_{*,L}(Y, \mathbb{C}) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L. \tag{2.2}$$

Setting $H^*_L(Y, \omega_Y) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L = H^*_{\mathcal{K}_L}(Y, \omega_Y)$ and $H^*_L(Y, \mathbb{C}) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L = H^*_{\mathcal{K}_L}(Y, \mathbb{C})$ we see that from (2.1) and (2.2) we get the natural pairing

$$H^*_{\mathcal{K}_L}(Y, \omega_Y) \underset{\mathcal{K}_L}{\otimes} H^*_{\mathcal{K}_L}(Y, \mathbb{C}) \to \mathcal{K}_L. \tag{2.3}$$

In particular, by pairing with the unit class we get the integration functional

$$\int_Y : H^*_{\mathcal{K}_L}(Y, \omega_Y) \to \mathcal{K}_L.$$

---

[4]Here and in what follows the subscript "c" denotes the appropriate cohomology with compact support

Since we have the obvious map $H_L^*(Y, \omega_Y) \to H_{\mathcal{K}_L}^*(Y, \omega_Y)$ the functional $\int_Y$ makes sense on $H_L^*(Y, \omega_Y)$ too (but of course it still takes values in $\mathcal{K}_L$). This functional is "homogeneous" in the following sense. Assume that we are given some $\eta \in H_L^i(Y, \omega_Y)$. If i is odd then $\int_Y \eta = 0$. If i is even then $\int_Y \eta$ considered as a rational function on $\mathfrak{l}$ is homogeneous of degree $\frac{i}{2}$.

It is easy to see that all of the above constructions make sense for an arbitrary reductive L provided that the scheme Y^T is proper where $T \subset L$ is a maximal torus.

2.4. Integration on ordinary cohomology and the integral of the unit class.

By composing $\int_Y$ with the map $H_L^*(Y, \mathbb{C}) \to H_L^{*-2\dim Y}(Y, \omega_Y)$ we may define the integration functional on $H_L^*(Y, \mathbb{C})$ (or $H_{\mathcal{K}_L}^*(Y, \mathbb{C})$). Abusing slightly the notation we shall denote this functional by the same symbol. This functional also vanishes on the odd part of $H_L^*(Y, \mathbb{C})$ and it sends any class $\psi \in H_L^{2d}(Y, \mathbb{C})$ to an element of $\mathcal{K}_L$ which (considered as a rational function on $\mathfrak{l}$) is homogeneous of degree $d - \dim Y$. In particular, we may consider the integral

$$\int_Y 1_Y$$

of the unit class $1_Y \in H_L^*(Y, \mathbb{C})$. This is a rational function on $\mathfrak{l}$ of degree $-\dim Y$.

Here is the basic example of the above situation. Assume that Y is smooth. Let $\mathfrak{t}$ denote the Lie algebra of T. Let also W_L denote the Weyl group of L. Then $\mathcal{A}_L = \mathcal{A}_T^{W_L}$ and $\mathcal{K}_L = \mathcal{K}_T^{W_L}$. For every $y \in Y^T$ let $d_y \in \mathcal{A}_T$ denote the determinant of the $\mathfrak{t}$-action on the cotangent space $T_y^* Y$ to Y at y. We have $d_y \neq 0$ if and only if y is an isolated fixed point. Assume that Y^T is finite. Then it is well known that

$$\int_Y 1_Y = \sum_{y \in Y^T} \frac{1}{d_y} \in \mathcal{K}_T^{W_L} = \mathcal{K}_L. \tag{2.4}$$

More generally, assume that both Y and Y^T are smooth. Let $c \in H^*(Y^T, \mathbb{C}) \otimes \mathcal{A}_T$ denote the equivariant Euler class of the conormal bundle to Y^T in Y. Then it is easy to see that c is invertible in $H^*(Y^T, \mathbb{C}) \otimes \mathcal{K}_T$ and we have

$$\int_Y 1_Y = \int_{Y^T} \frac{1}{c}.$$

In general, when Y is singular (and, say, Y^T is finite) the integral $\int_Y 1_Y$ can also be written as a sum over all $y \in Y^T$ of certain "local contributions" taking values in $\mathcal{K}_T$. However, in the general case there is no good formula for these local contributions. One of the purposes of this paper is to compute $\int_Y 1_Y$ in certain special cases.

The following lemma follows easily from the definitions and will be used several times in this paper:

Lemma 2.5. *Let $f : Y_1 \to Y_2$ be a proper birational map of L-varieties. Assume also that $\int_{Y_2} 1_{Y_2}$ is well-defined. Then $\int_{Y_1} 1_{Y_1}$ is also well-defined and we have*

$$\int_{Y_1} 1_{Y_1} = \int_{Y_2} 1_{Y_2}.$$

2.6. **The intersection cohomology approach.** We now want to give yet another definition of the above integration based on intersection cohomology. In fact, in order to do this we will need to make an additional assumption about the action of L on Y (which will be satisfied for the spaces ${}^{\mathrm{b}}\mathcal{QM}^{\theta}_{G,P}$ and $\mathcal{U}^{\theta_{\mathrm{aff}}}_{G,P}$ discussed in the introduction).

Let Y and L be as before and let also IC_Y denote the intersection cohomology complex of Y. Set

$$\mathrm{IH}^*_L(Y) = H^*_L(\mathrm{IC}_Y); \qquad \mathrm{IH}^*_{c,L} = H^*_{c,L}(\mathrm{IC}_Y). \tag{2.5}$$

Both spaces $\mathrm{IH}^*_L(Y)$ and $\mathrm{IH}^*_{c,L}(Y)$ are graded $\mathcal{A}_L$-modules. We have the natural equivariant Poincare pairing

$$\langle \cdot, \cdot \rangle_Y : \mathrm{IH}^*_L(Y) \underset{\mathcal{A}_L}{\otimes} \mathrm{IH}^*_{c,L}(Y) \to \mathcal{A}_L.$$

This pairing is homogeneous of degree $- \dim Y$. Also we have the natural (forgetting the supports) morphism

$$\mathrm{IH}^*_{c,L}(Y) \to \mathrm{IH}^*_L(Y)$$

of graded $\mathcal{A}_L$-modules.

Set $\mathrm{IH}_{\mathcal{K}_L}(Y) = \mathrm{IH}_L(Y) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L$ and $\mathrm{IH}_{c,\mathcal{K}_L}(Y) = \mathrm{IH}_{c,L}(Y) \underset{\mathcal{A}_L}{\otimes} \mathcal{K}_L$. Abusing slightly the notation we shall denote the corresponding pairing between $\mathrm{IH}_{c,\mathcal{K}_L}(Y)$ and $\mathrm{IH}_{\mathcal{K}_L}(Y)$ (which now takes values in $\mathcal{K}_L$) also by by $\langle \cdot, \cdot \rangle_Y$).

We have the natural morphisms $\mathbb{C}_Y[\dim Y] \to \mathrm{IC}_Y$ and $\mathrm{IC}_Y \to \omega_Y[-\dim Y]$ which give rise to the maps $H^{*+\dim Y}_L(Y, \mathbb{C}) \to \mathrm{IH}^*_L(Y)$ and $\mathrm{IH}^*_{c,L}(Y) \to H^{*-\dim Y}_{c,L}(Y, \omega_Y) = H_{*-\dim Y}(Y, \mathbb{C})$ which are compatible with the above pairings in the obvious sense. In particular we may consider the image of the unit cohomology class 1_Y in $\mathrm{IH}^*_L(Y)$ which we shall denote by the symbol.

Assume now that the map $\mathrm{IH}_{c,\mathcal{K}_L} \to \mathrm{IH}_{\mathcal{K}_L}$ is an isomorphism. Then we may consider the Poincaré pairing as a paring

$$\langle \cdot, \cdot \rangle_Y : \mathrm{IH}_{\mathcal{K}_L}(Y) \otimes \mathrm{IH}_{\mathcal{K}_L}(Y) \to \mathcal{K}_L.$$

Let $1_Y \in \mathrm{IH}_{\mathcal{K}_L}(Y)$ denote the unit cohomology class. For every $\omega \in \mathrm{IH}_{\mathcal{K}_L}$ we set

$$\int_Y \omega = \langle \omega, 1_Y \rangle_Y \in \mathcal{K}_L.$$

In particular one may consider the integral $\int_Y 1$.

Here is a condition which guarantees that the map $\mathrm{IH}_{c,\mathcal{K}_L} \to \mathrm{IH}_{\mathcal{K}_L}$ is an isomorphism.

Lemma 2.7. *Let Y be as above. Assume that*

a) Y^T is proper;

b) There exists an embedding $\mathbb{C}^* \subset T$ whose action on Y is contracting (this means that the action morphism $\mathbb{C}^* \times Y \to Y$ extends to a morphism $\mathbb{C} \times Y \to Y$.

Then the natural map $\mathrm{IH}_{c,\mathcal{K}_L} \to \mathrm{IH}_{\mathcal{K}_L}$ is an isomorphism.

In particular, under the assumptions of the lemma one may consider the integral $\int_Y 1 \in \mathcal{K}_L$. It is easy to see that this definition of $\int_Y 1$ coincides with the one discussed above.

3. THE MAIN RESULT

The purpose of this section is to formulate more carefully the results announced in the introduction.

In this section we are going to work over an arbitrary field k of characteristic 0 (later k will become the field $\mathcal{K}_T$ considered above).

Let $G, \mathfrak{g}$ be as in the introduction - i.e. $\mathfrak{g}$ is a simple Lie algebra and G is the corresponding simply connected algebraic group. The Lie algebra $\mathfrak{g}$ has its triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

Let $T \subset G$ be the corresponding Cartan subgroup. We denote by Λ_G the lattice of cocharacters of T. The lattice Λ_G is spanned by the coroots $\alpha : \mathbb{G}_m \to T$ and we shall refer to Λ_G as the coroot lattice. In fact, we let $\mathcal{I}_G$ denote the set of verties of the Dynkin diagram of G then Λ_G is spanned by the simple coroots $\alpha_{\mathfrak{i}}$ for $\mathfrak{i} \in \mathcal{I}_G$. We also let $\Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ denote the sub-semigroup whose elements are linear combinations of positive coroots with non-negative coefficients (equivalently, linear combinations of $\alpha_{\mathfrak{i}}$'s with non-negative coefficients).

Let $P \subset G$ be a parabolic subgroup with Levi component M. We denote by $\Lambda_{G,P}$ the lattice of cocharacters of the group $M/[M,M]$. We have the natural map $\Lambda_G \to \Lambda_{G,P}$ and we let $\Lambda_{G,P}^{\mathrm{pos}}$ denote the positive span of the images of $\alpha_{\mathfrak{i}}$'s in $\Lambda_{G,P}$ for $\mathfrak{i} \in \mathcal{I}_G \backslash \mathcal{I}_M$.

3.1. Verma modules and Whittaker vectors: the finite dimensional case.

Let $\check{\mathfrak{g}}$ denote the Langlands dual algebra of $\mathfrak{g}$. We have the triangular decomposition

$$\check{\mathfrak{g}} = \check{\mathfrak{n}}_+ \oplus \check{\mathfrak{h}} \oplus \check{\mathfrak{n}}_-.$$

Note that the set of vertices of the Dynkin diagram of $\check{\mathfrak{g}}$ is equal to $\mathcal{I}_G$. We denote by $e_{\mathfrak{i}} \in \check{\mathfrak{n}}_+$ and $f_{\mathfrak{i}} \in \check{\mathfrak{n}}_-$ (for $i \in \mathcal{I}_G$) the corresponding Chevalley generators of $\check{\mathfrak{g}}$. We denote by $\kappa : \check{\mathfrak{g}} \to \check{\mathfrak{g}}$ the Cartan anti-involution sending $e_{\mathfrak{i}}$ to $f_{\mathfrak{i}}$ and acting on $\check{\mathfrak{h}}$ as identity.

Recall that the lattice Λ_G defined above may be considered as the coroot lattice of $\check{\mathfrak{g}}$. Also we may think of $\mathfrak{h}$ as the space of all (not necessarily integral) weights of $\check{\mathfrak{g}}$. We let $\rho \in \mathfrak{h}$ denote the half-sum of the positive roots of $\check{\mathfrak{g}}$. Also we denote by $\check{\rho}$ the half-sum of the positive roots of $\mathfrak{g}$.

Choose now any $\lambda \in \mathfrak{h}$. We denote by $M(\lambda)$ the $\check{\mathfrak{g}}$-module generated by a vector v_λ such that:

1) $\check{\mathfrak{n}}_-$ annihilates v_λ;

2) For any $t \in \check{\mathfrak{h}}$ one has $t(v_\lambda) = \lambda(t)v_\lambda$.

3) v_λ is a free generator of $M(\lambda)$ over $U(\check{\mathfrak{n}}_+)$.

We shall call $M(\lambda)$ the *Verma module with lowest weight* λ. We also denote by $\delta M(\lambda)$ the corresponding dual Verma module; by the definition it consists of all $\check{\mathfrak{h}}$-finite linear functionals $m : M(\lambda) \to k$ with the $\check{\mathfrak{g}}$-action defined by

$$x(m)(v) = m(\kappa(x)(v))$$

for $x \in \check{\mathfrak{g}}, v \in M(\lambda), m \in \delta M(\lambda)$.

We have the unique map $\eta_\lambda : M(\lambda) \to \delta M(\lambda)$ of $\check{\mathfrak{g}}$-modules satisfying $\eta_\lambda(v_\lambda)(v_\lambda) = 1$. One can interpet this map as a pairing

$$\langle \cdot, \cdot \rangle : M(\lambda) \otimes M(\lambda) \to k$$

satisfying

$$\langle x(v), w \rangle = \langle v, \kappa(x)(w) \rangle$$

for each $x \in \check{\mathfrak{g}}$ and $v, w \in M(\lambda)$. It is well-known that for generic λ the map η_λ is an isomorphism. In fact for this it is enough to require, for example, that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all coroots $\alpha^\vee$ of $\check{\mathfrak{g}}$.

Let $\widehat{M}(\lambda)$ denote the completion of $M(\lambda)$ consisting of all sums

$$\sum m_\mu \qquad \text{where } m_\mu \text{ im } M(\lambda)_\mu$$

such that for all $\theta \in \Lambda_G^{\mathrm{pos}}$ the set

$$\{\mu | \ m_\mu \neq 0 \quad \text{and} \quad \mu \notin \lambda + \theta + \Lambda_G^{\mathrm{pos}}\}$$

is finite. It is clear that the Lie algebra $\check{\mathfrak{g}}$ still acts on $\widehat{M}(\lambda)$.

It is well-known that for any λ for which the map η_λ is an isomorphism and for every $\hbar \in k$ such that $\hbar \neq 0$ there exists unique vector $w(\lambda) \in \widehat{M}(\lambda), w(\lambda) = \sum w(\lambda)_\mu$ (where $w(\lambda)_\mu$ has weight μ) such that

1) $w(\lambda)_\lambda = v_\lambda$;

2) For any $\mathfrak{i} \in \mathcal{I}_G$ we have $f_\mathfrak{i}(w) = \frac{1}{\hbar} w$.

We shall refer to w as the *Whittaker vector* in $M(\lambda)$ (of course it depends on the choice of $\hbar$; however later this choice will become canonical).

We also set

$$w'(\lambda) = \sum_\mu (-1)^{\langle \lambda - \mu, \check{\rho} \rangle} w(\lambda)_\mu.$$

Clearly $w'(\lambda)$ satisfies the equation $f_\mathfrak{i}(w'(\lambda)) = -\frac{1}{\hbar} w'(\lambda)$ for all $\mathfrak{i} \in \mathcal{I}_G$; $w'(\lambda)$ is uniquely characterized by this property if in addition one requires that the λ-weight part of w' is equal to v_λ.

3.2. The affine case.

Let us denote by $\mathfrak{g}_{\mathrm{aff}}$ the full untwisted affine Lie algebra corresponding to $\mathfrak{g}$ (cf. [16]). As a vector space the Lie algebra $\mathfrak{g}_{\mathrm{aff}}$ can be written as

$$\mathfrak{g}_{\mathrm{aff}} = \mathfrak{g}[x, x^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where K is a central element of $\mathfrak{g}_{\mathrm{aff}}$ and $\mathrm{ad}(d)$ acts on $\mathfrak{g}[x, x^{-1}]$ as $x\frac{d}{dx}$. The Cartan subalgebra $\mathfrak{h}_{\mathrm{aff}}$ of $\mathfrak{g}_{\mathrm{aff}}$ is equal to $\mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We denote by $\mathcal{I}_G^{\mathrm{aff}}$ the set of vertices of the Dynkin diagram of $\mathfrak{g}_{\mathrm{aff}}$. Also we denote by $\check{\mathfrak{g}}^{\mathrm{aff}}$ the Langlands dual Lie algebra of $\mathfrak{g}_{\mathrm{aff}}$. By the definition this is an affine Lie algebra whose Cartan matrix is the

transposed of that of $\mathfrak{g}_{\mathrm{aff}}$; note that $\check{\mathfrak{g}}_{\mathrm{aff}}$ might be a twisted affine Lie algebra. For each $\mathfrak{i} \in \mathcal{I}_G^{\mathrm{aff}}$ we denote by $\alpha_{\mathfrak{i}}^{\mathrm{aff}}$ the corresponding simple root of the algebra $\check{\mathfrak{g}}_{\mathrm{aff}}$.

The Lie algebras $\mathfrak{g}_{\mathrm{aff}}$ and $\check{\mathfrak{g}}_{\mathrm{aff}}$ admit triangular decompositions

$$\mathfrak{g}_{\mathrm{aff}} = \mathfrak{n}_{+,\mathrm{aff}} \oplus \mathfrak{h}_{\mathrm{aff}} \oplus \mathfrak{n}_{-,\mathrm{aff}}$$

and

$$\check{\mathfrak{g}}_{\mathrm{aff}} = \check{\mathfrak{n}}_{+,\mathrm{aff}} \oplus \check{\mathfrak{h}}_{\mathrm{aff}} \oplus \check{\mathfrak{n}}_{-,\mathrm{aff}}.$$

Here $\mathfrak{h}_{\mathrm{aff}} = \mathfrak{h} \oplus \mathbb{C}d \oplus \mathbb{C}K$ and $\check{\mathfrak{h}}_{\mathrm{aff}} = \check{\mathfrak{h}} \oplus \mathbb{C}\check{d} \oplus \mathbb{C}\check{K}$. We have the natural perfect paring between $\mathfrak{h}_{\mathrm{aff}}$ and $\check{\mathfrak{h}}_{\mathrm{aff}}$.

Let $\mathfrak{h}'_{\mathrm{aff}} = \mathfrak{h} \oplus \mathbb{C}K$ and $\mathfrak{h}''_{\mathrm{aff}} = \mathfrak{h} \oplus \mathbb{C}d$. Similarly we define $\check{\mathfrak{h}}'_{\mathrm{aff}}$ and $\check{\mathfrak{h}}''_{\mathrm{aff}}$. It is well-known that the above perfect pairing identifies $\mathfrak{h}'_{\mathrm{aff}}$ with the dual space of $\check{\mathfrak{h}}''_{\mathrm{aff}}$; it also identifies $\mathfrak{h}''_{\mathrm{aff}}$ with the dual space of $\check{\mathfrak{h}}'_{\mathrm{aff}}$.

We also choose an affine analogue $\rho_{\mathrm{aff}} \in \mathfrak{h}''_{\mathrm{aff}}$ of ρ (cf. [16] for a detailed discussion).

Set $\check{\mathfrak{g}}'_{\mathrm{aff}}$ to be the subalgebra of $\check{\mathfrak{g}}_{\mathrm{aff}}$ consisting of elements whose projection to $\mathbb{C}\check{d}$ is equal to 0. We have

$$\check{\mathfrak{g}}' = \check{\mathfrak{n}}_{+,\mathrm{aff}} \oplus \check{\mathfrak{h}}'_{\mathrm{aff}} \oplus \mathfrak{n}_{-,\mathrm{aff}}.$$

For any $\lambda_{\mathrm{aff}} = (\lambda, \varepsilon) \in \mathfrak{h}''_{\mathrm{aff}} = (\check{\mathfrak{h}}'_{\mathrm{aff}})^*$ we let $M(\lambda_{\mathrm{aff}})$ denote the corresponding Verma module with lowest weight λ_{aff} over $\check{\mathfrak{g}}'_{\mathrm{aff}}$. In other words this module is freely generated by a vector $v_{\lambda_{\mathrm{aff}}}$ subject to the conditions:

1) $t(v_{\lambda_{\mathrm{aff}}}) = \lambda_{\mathrm{aff}}(t)v_{\lambda_{\mathrm{aff}}}$ for any $t \in \check{\mathfrak{h}}'_{\mathrm{aff}}$.

2) $v_{\lambda_{\mathrm{aff}}}$ is annihilated by $\mathfrak{n}_{+,\mathrm{aff}}$.

It is well-known that for generic ε (more precisely for all but one (critical) value of ε) the $\check{\mathfrak{g}}'_{\mathrm{aff}}$-module structure on $M(\lambda_{\mathrm{aff}})$ extends uniquely to an action of the whole $\check{\mathfrak{g}}_{\mathrm{aff}}$ such that d annihilates $v_{\lambda_{\mathrm{aff}}}$. Thus for such ε we may regard $M(\lambda_{\mathrm{aff}})$ as a $\check{\mathfrak{g}}_{\mathrm{aff}}$-module.

The Lie algebra $\check{\mathfrak{g}}_{\mathrm{aff}}$ carries canonical anti-involution κ_{aff} (interchanging $\check{\mathfrak{n}}_{+,\mathrm{aff}}$ and $\check{\mathfrak{n}}_{-,\mathrm{aff}}$ and acting as identity on $\check{\mathfrak{h}}_{\mathrm{aff}}$). Similarly to the affine case there is a unique pairing

$$\langle \cdot, \cdot \rangle : M(\lambda_{\mathrm{aff}}) \otimes M(\lambda_{\mathrm{aff}}) \to \mathbb{C}$$

satisfying

$$\langle x(v), w \rangle = \langle v, \kappa_{\mathrm{aff}}(x)(w) \rangle$$

for each $x \in \check{\mathfrak{g}}$ and $v, w \in M(\lambda_{\mathrm{aff}})$ and such that

$$\langle v_\lambda, v_\lambda \rangle = 1.$$

Of course all the above constructions make sense when the base field $\mathbb{C}$ is replaced by any field k of characteristic 0 (later k will become certain field of rational functions).

As before one may define a completion $\widehat{M}(\lambda_{\mathrm{aff}})$ of $M(\lambda_{\mathrm{aff}})$. Choose $\hbar \in k^*$. Then for generic λ_{aff} this completion will contain unique vector $w(\lambda_{\mathrm{aff}}) = \sum w(\lambda_{\mathrm{aff}})_\mu$ (here $\mu \in \mathfrak{h}''_{\mathrm{aff}}$) subject to the conditions

1) $w(\lambda_{\mathrm{aff}})_{\lambda_{\mathrm{aff}}} = v_{\lambda_{\mathrm{aff}}}$;

2) For any $\mathfrak{i} \in \mathcal{I}_G^{\mathrm{aff}}$ we have $f_{\mathfrak{i}}(w(\lambda_{\mathrm{aff}})) = \frac{1}{\hbar}w(\lambda_{\mathrm{aff}})$.

Again we shall call $w(\lambda_{\mathrm{aff}})$ the *Whittaker vector* of $M(\lambda_{\mathrm{aff}})$. Similarly one can define the vector $w'(\lambda_{\mathrm{aff}}$.

3.3. **The main result.** Set now

$$\mathrm{IH}_{G,P} = \mathrm{IH}_{\mathcal{K}_{M\times\mathbb{C}^*}}(\mathcal{QM}_{G,P}^{\theta}) \quad \text{and} \quad \mathrm{IH}_{G,P}^{\theta_{\mathrm{aff}}} = \mathrm{IH}_{\mathcal{K}_{M\times(\mathbb{C}^*)^2}}(\mathcal{U}_{G,P}^{\theta_{\mathrm{aff}}}).$$

Also we denote

$$\mathrm{IH}_{G,P} = \bigoplus_{\theta\in\Lambda_{G,P}^{\mathrm{pos}}} \mathrm{IH}_{G,P}^{\theta} \quad \text{and} \quad \mathrm{IH}_{G,P}^{\mathrm{aff}} = \bigoplus_{\theta_{\mathrm{aff}}\in\Lambda_{G,P}^{\mathrm{aff,pos}}} \mathrm{IH}_{G,P}^{\theta_{\mathrm{aff}}}.$$

We shall further restrict ourselves to the case $P = B$ (where B as before denotes a Borel subgroup of G). In this case $\mathrm{IH}_{G,B}$ is a vector space over the field $\mathcal{K}_{T\times\mathbb{C}^*}$ which we may identify with the field of rational functions on $\mathfrak{h}\times\mathbb{C}$. We shall denote a typical element in this product by $(a,\hbar)$. Similarly $\mathrm{IH}_{G,B}^{\mathrm{aff}}$ is a vector space over the field $\mathcal{K}_{T\times(\mathbb{C}^*)^2}$ which we may identify with the field of rational functions on $\mathfrak{h}\times\mathbb{C}^2$. We shall typically denote an element of the latter product as $(a,\varepsilon,\hbar)$. Note that we may think of (a,ε) as an element of $\mathfrak{h}'_{\mathrm{aff}}$. Note also that we have a canonical element $\hbar$ in both $\mathcal{K}_{T\times\mathbb{C}^*}$ and $\mathcal{K}_{T\times(\mathbb{C}^*)^2}$.

We denote by $\langle\cdot,\cdot\rangle_{G,B}^{\theta}$ (resp. by $\langle\cdot,\cdot\rangle_{G,B}^{\theta_{\mathrm{aff}}}$) the Poincare pairing on $\mathrm{IH}_{G,B}^{\theta}$ (resp. on $\mathrm{IH}_{G,B}^{\theta_{\mathrm{aff}}}$). As in the introduction we let $\langle\cdot,\cdot\rangle_{G,B}$ be the direct sum of the pairings $(-1)^{\langle\theta,\check{\rho}\rangle}\langle\cdot,\cdot\rangle_{G,B}^{\theta}$. Similarly we let $\langle\cdot,\cdot\rangle_{G,B}^{\mathrm{aff}}$ denote the direct sum of the pairings $(-1)^{\langle\theta_{\mathrm{aff}},\check{\rho}_{\mathrm{aff}}\rangle}\langle\cdot,\cdot\rangle_{G,B}^{\theta_{\mathrm{aff}}}$.

Theorem 3.4. *There exists a natural action of $\check{\mathfrak{g}}$ (resp. of $\check{\mathfrak{g}}_{\mathrm{aff}}$) on $\mathrm{IH}_{G,B}$ (resp. on $\mathrm{IH}_{G,B}^{\mathrm{aff}}$) such that:*

(1) *$\mathrm{IH}_{G,B}^{\theta}$ has weight $\frac{a}{\hbar}+\rho+\theta$ with respect to the action of $\check{\mathfrak{g}}$. Similarly, $\mathrm{IH}_{G,B}^{\theta_{\mathrm{aff}},\mathrm{aff}}$ has weight $\frac{(a,\varepsilon)}{\hbar}+\rho_{\mathrm{aff}}+\theta_{\mathrm{aff}}$ with respect to the action of $\check{\mathfrak{g}}_{\mathrm{aff}}$ [5].*

(2) *The pairing $\langle\cdot,\cdot\rangle_{G,B}$ is $(\check{\mathfrak{g}},\kappa)$-invariant. Similarly, the pairing $\langle\cdot,\cdot\rangle_{G,B}^{\mathrm{aff}}$ is $(\check{\mathfrak{g}}_{\mathrm{aff}},\kappa_{\mathrm{aff}})$-invariant.*

(3) *The $\check{\mathfrak{g}}$-module $\mathrm{IH}_{G,B}$ is isomorphic to $M(\frac{a}{\hbar}+\rho)$ as a $\check{\mathfrak{g}}$-module. Similarly, $\mathrm{IH}_{G,B}^{\mathrm{aff}}$ is isomorphic to $M_{\mathrm{aff}}(\frac{(a,\varepsilon)}{\hbar}+\rho_{\mathrm{aff}})$.*

(4) *Under the above identifications the vector $\sum 1_{\mathcal{QM}_{G,B}^{\theta}}$ (resp. $\sum 1_{\mathcal{U}_{G,B}^{\theta_{\mathrm{aff}}}}$) is the Whittaker vector $w(\frac{a}{\hbar}+\rho)$ (resp. $w(frac(a,\varepsilon)\hbar+\rho_{\mathrm{aff}}))$ in $\widehat{M}(\frac{a}{\hbar}+\rho)$ (resp. in $\widehat{M}(\frac{(a,\varepsilon)}{\hbar}+\rho_{\mathrm{aff}}))$.*

In what follows we let a_{aff} denote the pair (a,ε) (in the affine case).

Corollary 3.5. *We have*

$$\mathcal{Z}_{G,B} = \mathfrak{q}^{-\frac{a}{\hbar}-\rho}\langle w(\frac{a}{\hbar}+\rho), \mathfrak{q}\cdot w'(\frac{a}{\hbar}+\rho)\rangle \, [6]$$

---

[5] here we regard (a,ε) as an element of $\mathfrak{h}_{\mathrm{aff}}$ by means of the embedding $\mathfrak{h}'_{\mathrm{aff}}\subset\mathfrak{h}_{\mathrm{aff}}$

[6] Of course the torus $\check{T}$ does not act on $M(\frac{a}{\hbar})$ (only its Lie algebra acts). Thus in order to make sense of this formula we must (as before) write $q = e^t$ where $t\in\check{\mathfrak{h}}$ and thus the equality becomes an equality of formal Fourier series in t.

Similarly,

$$\mathcal{Z}_{G,B}^{\mathrm{aff}} = \mathfrak{q}^{-\frac{a}{\hbar}-\rho}\langle w(\frac{a_{\mathrm{aff}}}{\hbar}+\rho_{\mathrm{aff}}), \mathfrak{q}_{\mathrm{aff}} \cdot w'(\frac{a_{\mathrm{aff}}}{\hbar}+\rho_{\mathrm{aff}})\rangle.$$

3.6. **The quadratic Toda Hamiltonian.** Fix now an invariant quadratic form $(\cdot,\cdot)$ on $\mathfrak{g}$ so that long roots on $\check{\mathfrak{g}}$ have squared norm 2. Denote by Δ the corresponding Laplacian on $\check{\mathfrak{h}}$. As before we set $\mathfrak{q} = e^t$ where $t \in \check{\mathfrak{h}}$ and $\mathfrak{q}_{\mathrm{aff}} = e^{t_{\mathrm{aff}}}$. Thus we may write $t_{\mathrm{aff}} = (t, \ln Q)$.

Corollary 3.7. (1) *The function $e^{\frac{a(t)}{\hbar}}\mathcal{Z}_{G,B}$ is an eigen-function of the quantum Toda Hamiltonians with the "eigen-value" defined by a (cf. [7] and references therein for the definition). In particular, the function $\mathcal{Z}_{G,B}$ satisfies the equation*

$$(\hbar^2\Delta - 2\sum_{\mathrm{i}\in\mathcal{I}_G} e^{\alpha_{\mathrm{i}}(t)})(e^{\frac{a(t)}{\hbar}}\mathcal{Z}_{G,B}) = (a,a)e^{\frac{a(t)}{\hbar}}\mathcal{Z}_{G,B}$$

(2) *The function $\mathcal{Z}_{G,B}^{\mathrm{aff}}$ satisfies*

$$(\varepsilon Q\frac{\partial}{\partial Q} + \hbar^2\Delta - 2\sum_{\mathrm{i}\in\mathcal{I}_G^{\mathrm{aff}}} e^{\alpha_{\mathrm{i}}^{\mathrm{aff}}(t_{\mathrm{aff}})})e^{\frac{a(t)}{\hbar}}\mathcal{Z}_{G,B}^{\mathrm{aff}} = (a,a)e^{\frac{a(t)}{\hbar}}\mathcal{Z}_{G,B}. \qquad (3.1)$$

The operator in the left hand side of (3.1) may be called a *non-stationary analogue* of the quadratic affine Toda hamiltonian (the latter will be obtained from it by removing the $\varepsilon Q\frac{\partial}{\partial Q}$-term).

4. EXAMPLE: $G = SL(2)$

In this section we would like to prove Corollary 3.5 explicitly for $G = SL(2)$ (in the non-affine case). So, we shall assume that $G = SL(2)$ until the end of this section.

4.1. **The space $^{\mathrm{b}}\mathcal{M}_{G,B}^d$.** Since $G = SL(2)$ we may identify Λ_G with $\mathbb{Z}$ and we shall typically denote an element of Λ_G by d (instead of θ). Also in this case $\mathcal{G}_{G,B} = \mathbb{P}^1$ (we choose this identification in such a way that $e_{G,B}$ corresponds to $\infty \in \mathbb{P}^1$) and thus $^{\mathrm{b}}\mathcal{M}_{G,B}^d$ is just the space of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d which send ∞ to ∞. This is the same as the set of pairs (g,h) of polynomials in one variable such that:

1)g and h have no common roots
2)$g(z) = z^d +$ lower order terms
3)$\deg h < d$.

4.2. **The space $^{\mathrm{b}}\mathcal{QM}_{G,B}^d$ and the integral of** 1. The "zastava" space $^{\mathrm{b}}\mathcal{QM}_{G,B}^d$ consists of all pairs g,h as above but with condition 1 removed (in particular h is allowed to be identically 0). Thus $^{\mathrm{b}}\mathcal{QM}_{G,B}^d$ is isomorphic to the affine space $\mathbb{A}^{2d}$. It inherits the natural action of $T \times \mathbb{C}^*$ with unique fixed point corresponding to $g = z^d, h = 0$.

Now for each d we set

$$A_d := \int_{^{\mathrm{b}}\mathcal{QM}_{G,B}^d} 1 \quad \in \mathcal{K}_{T\times\mathbb{C}^*}$$

(the integral is taken in the $T \times \mathbb{C}^*$-equvariant cohomology). Note that $T = \mathbb{C}^*$ in our case; thus we shall identify $\mathcal{K}_{T \times \mathbb{C}^*}$ with the field of rational functions in the variables $a, \hbar$. Since $\mathcal{M}^d$ is smooth and we have just one fixed point it is easy to compute this integral explicitly by using (2.4) and we get

$$A_d = \frac{1}{d! \hbar^d \prod\limits_{i=1}^{d} (a + i\hbar)}.$$

4.3. **Verma modules and Whittaker vectors.** Let us denote by e, h, f the standard basis for $sl(2) \simeq \check{\mathfrak{g}}$. For every scalar λ we denote by $M(\lambda)$ (as before) the Verma module over $sl(2)$ with *lowest weight* λ. Explicitly this means that $M(\lambda + 1)$ has a basis $\{m_d\}_{d \geq 0}$ in which the action is written in the following way:

$$h(m_d) = (\lambda + 2d + 1)m_d; \quad e(m_d) = m_{d+1}; \quad f(m_d) = -d(\lambda + d)m_{d-1}$$

(thus f acts nilpotently and $\mathbb{C}[e]$ acts freely).

The anti-involution $\kappa : sl(2) \to sl(2)$ is written in the following way:

$$\kappa(e) = f; \quad \kappa(f) = e \quad \kappa(h) = h.$$

Recall that $\langle \cdot, \cdot \rangle$ denotes the unique $(sl(2), \kappa)$-invariant form on $M(\lambda + 1)$ satisfying $\langle m_0, m_0 \rangle = 1$.

Explicitly this form is written in the following way:

$$\langle m_d, m_{d'} \rangle = \delta_{d,d'} (-1)^d d! \prod_{i=1}^{} (\lambda + i).$$

We now consider the case $\lambda = \frac{a}{\hbar}$. In this case define

$$w_d = \frac{v_d}{d! \prod\limits_{i=1}^{d} (a + i\hbar)}.$$

Thus

$$f(w_d) = -\frac{1}{\hbar} w_{d-1}.$$

Set also $w = \sum (-1)^d w_d$, $w' = \sum w_d$. Thus $f(w) = \frac{1}{\hbar} w$ and $f(w') = -\frac{1}{\hbar} w'$ - i.e. $w = w(\frac{a}{\hbar} + 1)$ and $w' = w'(\frac{a}{\hbar} + 1)$.

On the other hand we have

$$\langle w_d, w_d \rangle = \frac{\langle v_d, v_d \rangle}{(d! \prod\limits_{i=1}^{d} (a + i\hbar))^2} = \frac{(-1)^d d! \prod\limits_{i=1}^{d} (\frac{a}{\hbar} + i)}{(d! \prod\limits_{i=1}^{d} (a + i\hbar))^2} = \frac{(-1)^d}{d! \hbar^d \prod\limits_{i=1}^{d} (a + i\hbar)} = (-1)^d A_d.$$

$$(4.1)$$

This is equivalent to Corollary 3.5 in this case.

4.4. **The geometric interpretation of the action.** To complete the picture for $G = SL(2)$ let us explain the geometric origin of the operators (e, h, f) on $\mathrm{IH}_{G,B}$. Since e is adjoint to f with respect to $\langle \cdot, \cdot \rangle$ and $[e, f] = h$ it follows that it is enough to give a construction of f.

For each $d \geq 0$ let $Y_d \subset {}^b\mathcal{QM}_{G,B}^d \times {}^b\mathcal{QM}_{G,B}^{d+1}$ consisting of pairs $((g_1, h_1), (g_2, h_2))$ subject to the following condition:

there exists $z_0 \in \mathbb{C}$ so that $g_2(z) = (z - z_0)g_1(z)$ and $h_2(z) = (z - z_0)h_1(z)$.

We denote by pr_1 the projection from Y_d to ${}^b\mathcal{QM}_{G,B}^d$ and by pr_2 the corresponding projection to ${}^b\mathcal{QM}_{G,B}^{d+1}$.

The variety Y_d viewed as a correspondence from ${}^b\mathcal{QM}_{G,B}^{d+1}$ to ${}^b\mathcal{QM}_{G,B}^d$ defines a $\mathcal{K}_{T \times \mathbb{C}^*}$-linear map $H^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^{d+1}) \to H^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d)$. Since we are ignoring the grading and since in this case the varieties ${}^b\mathcal{QM}_{G,B}^d$ are smooth we have $H^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d) = \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d)$ and thus we let $f : \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^{d+1}) \to \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d)$ be the operator defined by Y_d as a correspondence. Since Y_d is clearly isomorphic to ${}^b\mathcal{QM}_{G,B}^d \times \mathbb{A}^1$ to see that $f(1_{G,B}^{d+1}) = \frac{1_{G,B}^d}{\hbar}$.

We now define $h : \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d) \to \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d)$ to be the operator of multiplication by $\frac{a}{\hbar} + 2d + 1$ and $e : \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}{}^b(\mathcal{QM}_{G,B}^d) \to \mathrm{IH}^*_{\mathcal{K}_{T \times \mathbb{C}^*}}({}^b\mathcal{QM}_{G,B}^d)$ to be the conjugate of f with respect to the pairing $\langle \cdot, \cdot \rangle_{G,B}$.

We now have to check that the operators e, h, f defined in the above way satisfy the relations of the Lie algebra $sl(2)$. The relations $[h, e] = 2e$ and $[h, f] = -2f$ are obvious. Thus we have to check only that $[e, f] = h$. This is equivalent to (4.1) (since $A_d = (-1)^d \langle 1_{G,B}^d, 1_{G,B}^d \rangle_{G,B})$.

5. CONSTRUCTION OF THE ACTION

In this section we sketch the proof of Theorem 3.4 assuming that the reader is familiar with the contents of [8].

5.1. **Plan of the proof.** First we are going to discuss the finite case.

Recall that $\check{\mathfrak{g}}$ has the triangular decomposition

$$\check{\mathfrak{g}} = \check{\mathfrak{n}}_+ \oplus \check{\mathfrak{h}} \oplus \check{\mathfrak{n}}_-.$$

The construction of the $\check{\mathfrak{g}}$-action on $\mathrm{IH}_{G,B}$ will consist of the following 3 steps:

• Step 1: Construct an action of $\check{\mathfrak{n}}_+$ on $\mathrm{IH}_{G,B}$.

• Step 2: Use the pairing $\langle \cdot, \cdot \rangle_{G,B} : \mathrm{IH}_{G,B} \underset{\mathcal{K}_{T \times \mathbb{C}^*}}{\otimes} \mathrm{IH}_{G,B} \to \mathcal{K}_{T \times \mathbb{C}^*}$ to construct the action of $\check{\mathfrak{n}}_-$.

• Step 3: Check that the two actions of $\check{\mathfrak{n}}_+$ and of $\check{\mathfrak{n}}_-$ generate an action of $\check{\mathfrak{g}}$ and that the resulting $\check{\mathfrak{g}}$-module is isomorphic to $M(\frac{a}{\hbar} + \rho)$.

• Step 4: Check that for every Chevalley generator f_i corresponding to some $i \in \mathcal{I}_G$ and every $\theta \in \Lambda_G^{\mathrm{pos}}$ we have

$$f_i(1_{G,B}^\theta) = \frac{1}{\hbar} 1_{G,B}^{\theta - \alpha_i}. \tag{5.1}$$

In fact steps 1-3 come from almost a word-by-word repetition of the contents of Section 4 of [8] and step 4 is equivalent to the computations of the previous section.

5.2. **Step 1.** Recall the definition of the *local Ext-algebra* from [8]. For every $\theta \in \Lambda_G^{\text{pos}}$ we denote (following [8]) by $\mathbb{A}^\theta$ the corresponding partially symmetrized power of $\mathbb{A}^1$ (i.e. elements of A^θ are formal sums $\sum \theta_i x_i$ with $\theta_i \in \Lambda_G^{\text{pos}}$, $x_i \in \mathbb{A}^1$ and $\sum \theta_i = \theta$). In [8] the authors construct a closed embedding $\mathbb{A}^\theta \hookrightarrow {}^{\text{b}}\mathcal{QM}_{G,B}^\theta$ whose image is denoted by $\partial^\theta({}^{\text{b}}\mathcal{QM}_{G,B}^\theta)$ (in fact this is the closed stratum of a certain canonical stratification of ${}^{\text{b}}\mathcal{QM}_{G,B}^\theta$).

Now we set (cf. Section 2 of [8])

$$\mathcal{A}_\theta = \text{Ext}_{{}^{\text{b}}\mathcal{QM}_{G,B}^\theta}^* (\text{IC}_{\partial^\theta({}^{\text{b}}\mathcal{QM}_{G,B}^\theta)}, \text{IC}_{{}^{\text{b}}\mathcal{QM}_{G,B}^\theta})$$

and define

$$\mathcal{A} = \bigoplus_{\theta \in \Lambda_G^{\text{pos}}} \mathcal{A}_\theta.$$

In [8] the authors define in a geometric way a Hopf algebra structure on $\mathcal{A}$ and show that it is isomorphic (as a Hopf algebra) to $U(\check{\mathfrak{n}}_+)$.

In section 4 of [8] the authors define an action of $\mathcal{A}$ on $\oplus_{\theta \in \Lambda_G^{\text{pos}}} \text{IH}^*(\mathcal{QM}_{G,B}^\theta)$. A word-by-word repetition of that construction defines an action of $\mathcal{A}$ on $\text{IH}_{G,B}$. Thus we get an action of the Lie algebra $\check{\mathfrak{n}}_+$ on $\text{IH}_{G,B}$.

5.3. **Steps 2,3 and 4.** We already have the action of the operators f_i on $\text{IH}_{G,B}$. We define the action of every $t \in \check{\mathfrak{h}}$ on $\text{IH}_{G,B}$ by requiring that t acts on $\text{IH}_{G,B}^\theta$ by means of multiplication by $\langle t, \frac{a}{\hbar} + \rho + \theta \rangle$ on $\text{IH}_{G,B}^\theta$. Also we define e_i to be the adjoint operator to f_i with respect to $\langle \cdot, \cdot \rangle_{G,B}$. We need to check the following relations:

$$[e_i, f_i] = h_i; \quad [e_i, f_j] = 0 \quad \text{for } i \neq j$$

The first relation is proved exactly as the Proposition in section 4.9 of [8] and the second one is proved exactly as Proposition 4.8 of [8].

In order to check that $\text{IH}_{G,B}$ is isomorphic to $M(\frac{a}{\hbar} + \rho)$ as a $\check{\mathfrak{g}}$-module we may argue (for example) as follows. It follows from the main theorem of section 3 of [8] (cf. section 3.1 of [8]) that $\text{IH}_{G,B}$ and $M(\frac{a}{\hbar} + \rho)$ have the same character. Also, it is clear that $\text{IH}_{G,B}$ lies in the category $\mathcal{O}$ for $\check{\mathfrak{g}}$ (in other words, $\check{\mathfrak{h}}$ acts on it in a semi-simple way with finite-dimensional weight spaces and $\mathfrak{n}_-$ acts locally nilpotetnly). However, it is easy to see that $M(\frac{a}{\hbar} + \rho)$ is the only module in this category with such character (in fact, it is the unique module in the category $\mathcal{O}$ with the lowest weight $\frac{a}{\hbar} + \rho$).

Finally, we need to check (5.1). However, using the same argument as in 4.9 of [8] we can reduce it to the case $G = SL(2)$ and thus it becomes equivalent to the calculation of the previous section.

5.4. **The affine case.** Let us explain how to extend the above argument to the affine case. Here we need to use the results of [3] instead of [8]. First of all the algebra $\mathcal{A}$ is defined exactly as above using the varieties $\mathcal{U}_{G,B}^{\theta_{\text{aff}}}$ instead of ${}^{\text{b}}\mathcal{QM}_{G,B}^\theta$ (one has to use the stratification of $\mathcal{U}_{G,B}^{\theta_{\text{aff}}}$ defined in section 10 of [3]). The definition of the Hopf algebra structure is a word-by-word repetition of the construction in section 2 of [8].

The proof of the isomorphism $\mathcal{A} \simeq U(\check{\mathfrak{n}}_{+,\mathrm{aff}})$ is the same as in section 4 of [8] except that one has to use the results of section 16 of [3] for counting the dimension of $\mathcal{A}_\theta$ instead of section 3 of [8]. The rest is proved exactly as before (except that in order to prove the isomorphism $\mathrm{IH}^{\mathrm{aff}}_{G,B} \simeq M(\frac{a_{\mathrm{aff}}}{\hbar} + \rho_{\mathrm{aff}})$ one has to use again section 16 of [3] instead of section 3 of [8] while computing the character of $\mathrm{IH}^{\mathrm{aff}}_{G,B}$).

6. Proof of Theorem 1.7

The contents of this section are mostly due to A. Givental. We include it for the sake of completeness.

6.1. **Some reformulations.** In this section we prove Theorem 1.7. The arguments here are very close to those in [14]. Fix G, P as in the formulation of the theorem. We need to prove that for each β we have

$$\int_{{}^{\mathrm{b}}\mathcal{QM}^\beta_{G,P}} 1^\beta_{G,P} = (\mathrm{ev}_\beta)_* \left(\frac{1}{\hbar(c_\beta + \hbar)} \right)$$

under the conventions explained before the formulation of Theorem 1.7. Let ${}^{\mathrm{b}}\overline{\mathcal{M}}^\beta_{0,1}(G, P)$ denote the space of based stable maps from a genus 0 curve with one marked point to $\mathcal{G}_{G,P}$; in other words ${}^{\mathrm{b}}\overline{\mathcal{M}}^\beta_{0,1}(G, P)$ is a closed sub-scheme in $\overline{\mathcal{M}}^\beta_{0,1}(G, P) := \overline{\mathcal{M}}^\beta_{0,1}(\mathcal{G}_{G,P})$ corresponding to maps sending the marked point to $e_{G,P}$.

It is now clear that the restriction map gives rise to an isomorphism $H^*_G(\overline{\mathcal{M}}^\beta_{0,1}(G, P), \mathbb{C}) \simeq H^*_M({}^{\mathrm{b}}\overline{\mathcal{M}}^\beta_{0,1}(G, P), \mathbb{C})$. Moreover, the following diagram commutes:

$$
\begin{array}{ccc}
H^*_G(\overline{\mathcal{M}}^\beta_{0,1}(G, P), \mathbb{C}) & \longrightarrow & H^*_M({}^{\mathrm{b}}\overline{\mathcal{M}}^\beta_{0,1}(G, P), \mathbb{C}) \\
{\scriptstyle (\mathrm{ev}_\beta)_*} \downarrow & & \downarrow {\scriptstyle \int_{{}^{\mathrm{b}}\overline{\mathcal{M}}^\beta_{0,1}(G,P)}} \\
H^*_G(\mathcal{G}_{G,P}, \mathbb{C}) & \longrightarrow & H^*_M(pt, \mathbb{C})
\end{array}
$$

where the last map is just the restriction to $e_{G,P}$.

In particular we see that under the above identifications we have

$$(\mathrm{ev}_\beta)_* \left(\frac{1}{\hbar(c_\beta + \hbar)} \right) = \int_{{}^{\mathrm{b}}\overline{\mathcal{M}}(G,P)} \frac{1}{\hbar(c_\beta|_{{}^{\mathrm{b}}\overline{\mathcal{M}}(G,P)} + \hbar)}.$$

Our next goal will be to rewrite the last integral as the integral of 1 over certain space.

6.2. **Graph spaces.** For every $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ let us introduce the *graph* spaces

$$\mathrm{Graph}^\beta_{G,P} := \overline{\mathcal{M}}^{\beta,1}_{0,0}(\mathcal{G}_{G,P} \times \mathbb{P}^1).$$

A point of $\mathrm{Graph}^\beta_{G,P}$ is a stable map $C \to \mathcal{G}_{G,P} \times \mathbb{P}^1$ which has degree θ over the first multiple and degree one over the second one. In the case when C is non-singular the second projection gives an isomorphism between C and $\mathbb{P}^1$ and thus our map is actually equal to the graph of some map $\mathbb{P}^1 \to \mathcal{G}_{G,P}$ of degree β. Thus $\mathcal{M}^\beta_{G,P}$ is an open subset of $\mathrm{Graph}^\beta_{G,P}$.

We denote by $^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}$ the corresponding "based" version of the graph space; it consists of all stable maps $C \to \mathcal{G}_{G,P}$ such that

(i) over some neighbourhood of $\infty \in \mathbb{P}^1$ the projection $C \to \mathbb{P}^1$ is an isomorphism. In this case we denote by ∞_C the (unique) preimage of ∞ in C.

(ii) The image of ∞_C in $\mathcal{G}_{G,P} \times \mathbb{P}^1$ is $(e_{G,P}, \infty)$.

Note that $^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}$ is not a closed subset of $\operatorname{Graph}_{G,P}^{\beta}$ (it is only locally closed).

6.3. The $\mathbb{C}^*$-action.
The group $\mathbb{C}^*$ acts naturally on $\operatorname{Graph}_{G,P}^{\beta}$ (via its action on $\mathbb{P}^1$). The fixed points of this action are discussed in [14]; it is shown in *loc. cit.* that we have

$$(\operatorname{Graph}_{G,P}^{\beta})^{\mathbb{C}^*} = \bigcup_{\beta_1+\beta_2=\beta} \overline{\mathcal{M}}_{0,1}^{\beta_1}(G,P) \times \overline{\mathcal{M}}_{0,1}^{\beta_2}(G,P)$$

This isomorphism is defined as follows. Assume that we have a stable map $f : C \to \mathcal{G}_{G,P} \times \mathbb{P}^1$ of degree $(\beta,1)$. Then there exists unique component C' of C which maps isomorphically to the second multiple. Assume now that f is $\mathbb{C}^*$-invariant. Then the composition $p_1 \circ f|_{C'}$ must be constant (here p_1 is the projection $\mathcal{G}_{G,P} \times \mathbb{P}^1 \to \mathcal{G}_{G,P}$). Let C'' be the union of all the irreducible components of C which are different from C'. Then the composition map $C'' \to \mathcal{G}_{G,P} \times \mathbb{P}^1 \to \mathbb{P}^1$ sends every connected component of C'' either to 0 or to ∞. We set $C_0'' = (p_2 \circ f|_{C''})^{-1}(0)$ and $C_\infty'' = (p_2 \circ f|_{C''})^{-1}(\infty)$. We also set $c_0 = C_0'' \cap C'$ and $c_\infty = C_\infty'' \cap C'$. Thus one can write $C'' = C_0'' \cup C_\infty''$ and the restriction of $p_1 \circ f$ to either (C_0'', c_0) or (C_∞'', c_∞) is a stable map. In particular, if $(C,f) \in {}^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}$ then $C'' = C_0''$ (since the unique point of C' lying over $\infty \in \mathbb{P}^1$ must be a non-singular point of C) and the map $(C,f) \mapsto (C'', c_0, p_1 \circ f|_{C''})$ defines an isomorphism

$$({}^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta})^{\mathbb{C}^*} \simeq {}^{\mathrm{b}}\overline{\mathcal{M}}_{0,1}^{\beta}(G,P)$$

Moreover, it is easy to see (cf. [14], proof of Theorem 2 or [12]) that the conormal bundle to $^{\mathrm{b}}\overline{\mathcal{M}}_{0,1}^{\beta}(G,P)$ in $^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}$ is of rank 2 and its $\mathbb{C}^*$-equivariant first Chern class is $\hbar(c_\beta + \hbar)$. Therefore we have

$$\int_{{}^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}} 1 = \int_{{}^{\mathrm{b}}\overline{\mathcal{M}}_{0,1}^{\beta}(G,P)} \frac{1}{\hbar(c_\beta + \hbar)} \tag{6.1}$$

6.4. End of the proof.
It follows from (6.1) that in order to prove Theorem 1.7 it is enough to show that

$$\int_{{}^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta}} 1 = \int_{{}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\beta}} 1 \tag{6.2}$$

For this it is enough to show the existence of a proper birational M-equivariant map $^{\mathrm{b}}\operatorname{Graph}_{G,P}^{\beta} \to {}^{\mathrm{b}}\mathcal{QM}_{G,P}^{\beta}$. This is proved in [8].

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